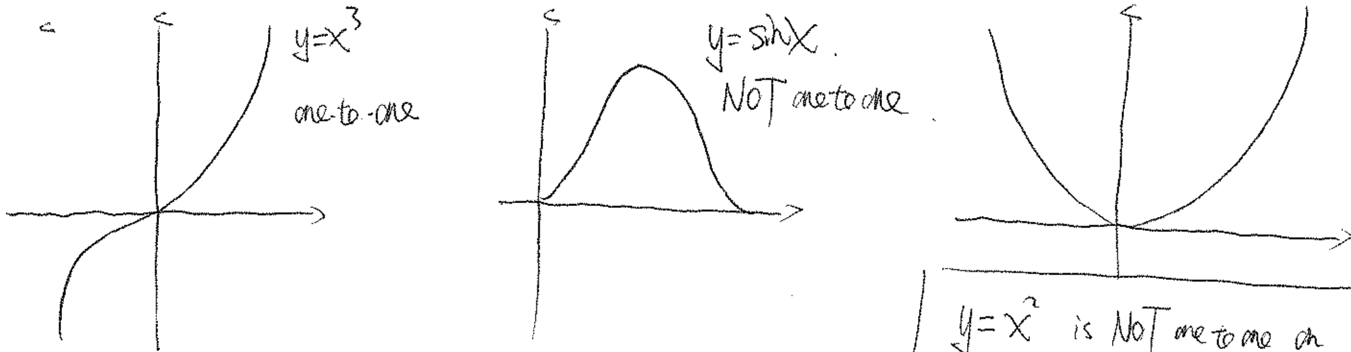
§6.1. Inverse function

- $f(x)$ is ONE-TO-ONE if $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2$.

  Equivallent to: No horizontal line intersects the graph of $f(x)$.

eg.1



$y=x^2$ is NOT one to one on $\mathbb{R}$

is one to one on $[0,+\infty)$

- Give a one-to-one function $y=f(x)$ (with domain A and range B)

  Solve the equation $y=f(x)$ for x (in terms of y).

  The solution is called THE INVERSE FUNCTION of $f(x)$.

  Switch x and y in the new function and denote it by $\boxed{y=f^{-1}(x)}$

- Properties of f^{-1}: Domain f^{-1} = Range f. Range f = Domain f^{-1}.

  $$f(f^{-1}(\square)) = \square \ ; \quad f^{-1}(f(\triangle)) = \triangle$$

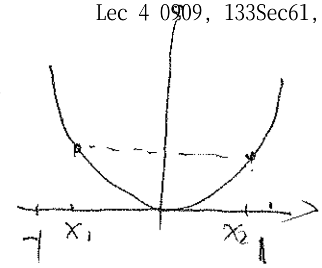
  The graph of f and f^{-1} $\boxed{\text{are SYMMETRIC with respect to } y=x}$

- KEY FORMULA for derivative of f^{-1} at the point $x=a$

☆ $$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

e.g. 2. • $y = x^2$ is NOT ONE-TO-ONE on $[-1, 1]$ since

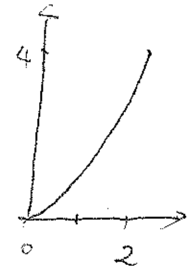
so has no inverse function on $[-1,1]$.



• $y = x^2$ is ONE-TO-ONE on $[0, 2]$

Domain : $[0, 2]$ (where x lives)

Range : $[0, 4]$. (where y lives)



(Three Steps to find the inverse of $y = x^2$)

Step 1: Write $y = x^2$

Step 2: Solve ~~for~~ x as a function of y: $x = \sqrt{y}$

Step 3: Interchange x and y in step 2

$x = \sqrt{y} \longrightarrow \boxed{y = \sqrt{x}.}$

(Caution: Actually, there are two solutions: $x = \pm\sqrt{y}$. We drop $-\sqrt{y}$ since $x \in [0,2]$ is positive)

Conclusion: The inverse function of $y = x^2$ on $[0, 2]$ is

$y = \sqrt{x}$, whose domain is $[0, 4]$ (the range of $y = x^2$)

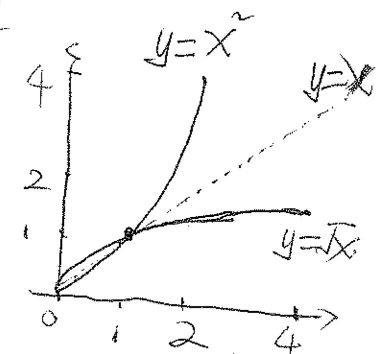
and whose range is $[0, 2]$ (the domain of $y = x^2$)

e.g. 3 : check eg 2 satisfies all properties listed on Page 1

$(f(f^{-1}(\square)) = \square \checkmark :)$ $\quad (\sqrt{\square})^2 = \square$

$(f^{-1}(f(\triangle)) = \triangle \checkmark :)$ $\quad \sqrt{x^2} = x$.

Graph:



e.g. 4 : Compute the derivatives of $y = x^2$ and $y = \sqrt{x}$. at $x = 4$.

Compare with ✪ formula at $x = 2$

$(x^2)'|_{x=2} = 2x|_{x=2} = 4$; $(\sqrt{x})'|_{x=4} = \frac{1}{2}\frac{1}{\sqrt{x}}|_{x=4} = \frac{1}{4}$.

Remark 1: The most important types of inverse functions will be dissussed in §6.2-6.7, which are log-exp, inverse-trig, inverse-hyp.

$\boxed{\text{Remark 2:}}$ Most functions do not have an explicit inverse function as in eg 2. But we can still ~~comp~~ study the derivative via $\boxed{\text{formula } \star}$.

eg 5: Let $f(x) = 2x^4 + 3x - 5$ for $x > 0$. Find $(f^{-1})'(x)$ at the point $x = 0 = f(1)$

(s16, mid)

s/n: (Step 1:) Compute the derivative of $f(x)$.

$$f'(x) = 2 \cdot 4 x^3 + 3$$

(Step 2:) Evaluate $f'(x)$ (at the CORRECT POINT $f^{-1}(a)$.)

In this example, $a = x = 0$. $\boxed{f(1) = 0 \Rightarrow 1 = f^{-1}(0)}$

Therefore, $f^{-1}(0) = 1$

and $f'(f^{-1}(0)) = f'(1) = 2 \cdot 4 \cdot 1^3 + 3 = 11$

(Step 3:) Flip (via $\star$)

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \boxed{\frac{1}{11}}$$

※

☆☆. eg 6. $y = f(x) = \sin x$ Compute the derivative of $\sin x$ (which is denoted by $\sin^{-1}$) at the point ~~$x = 0 \ \sin$~~ $x = a = \sin b$. In TERMS OF a.

s/n: S1: $f'(x) = (\sin x)' = \cos x$; S2: $a = \sin b$, i.e., $f(b) = a$

i.e. $b = f^{-1}(a)$

S2: $f'(f^{-1}(a)) = f'(b) = \cos b$.

S3: $(f^{-1})'(a) = \dfrac{1}{\cos b}$

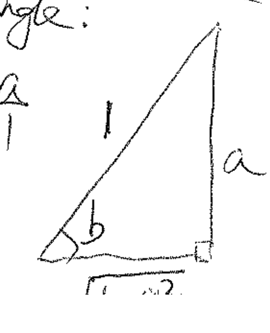
Extra Step: express $\cos b$ in terms of a

i.e. $\boxed{(f^{-1})'(a) = \dfrac{1}{\cos b} = \dfrac{1}{\sqrt{1-a^2}}}$

Solving Triangle:

$\sin b = a = \dfrac{a}{1}$

$\cos b = \sqrt{1-a^2}$



§6.2-6.4. (Natural) Logarithm / Exponential Functions and their Applications.

- log $\boxed{y=\log_a x}$ $\boxed{y=a^x}$ exponential

  nature log $\boxed{y=\ln x}$ $\boxed{y=e^x}$ natural exponential.

- Motivation: Generalization of integer power and its reverse

2^n: $2^3 = 2\times2\times2 = 8$, $2^4 = 2\times2\times2\times2 = 16$ $\Rightarrow 2^{3.5}$? $\xrightarrow{exp}$ 2^x for any x.

Reverse (Inverse)

$3 \longmapsto 8$, $4 \xmapsto{exp} 16$ reverse: $8 \longmapsto 3$, $16 \xmapsto{log} 4$ $\log_2 x$.

$2 \mapsto a$ $(a>0)$. $\boxed{y=a^x}$ Special case: $a=e$, $\boxed{y=e^x}$

exp-function with base a. natural exp.

Inverse

$\boxed{y=\log_a x}$ Special case: $a=e$, $\boxed{y=\ln x}$

log-function with base a natural log.

- Graph of a^x and $\log_a x$.

$a>1$: $0<a<1$